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Generalized Funk–Hecke theorem and non-local O(1, f) symmetric potentials

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Abstract. Introducing generalized Mehler and Fourier transforms, we extend the Funk–Hecke theorem to the case of non-compact O(1, f) groups, and we give the exact expressions for the energy spectrum and the wavefunctions of the Schrödinger equation with non-local O(1, f) symmetric potentials in the scattering region.

1. Introduction

Non-local potentials in the Schrödinger equation have been the subject of considerable investigation because of their phenomenological interest in nuclear and in atomic physics (Yamaguchi 1954, Perey and Buck 1962, Arnold and Mackellar 1971, Ali *et al* 1974), as well as because of their theoretical interest in nuclear theory (Wheeler 1937, Bethe 1956, Zohni 1973) and in connection with the extension of the results of local potentials (Gourdin and Martin 1957, 1958, Bertero *et al* 1968a, b, Gutkowski and Scalia 1968, 1969, Yao Hai Te 1973, 1974, Ahmad 1974). Because of the mathematical difficulties of the problem, non-local potentials of separable form and their simple generalizations are usually considered in order to obtain exact solutions of the Schrödinger equation. Another way to reduce the mathematical difficulties and obtain again exact solutions for the energy and the wavefunctions, in both the bound-state region and the scattering region, is to introduce symmetry into the problem.

Having exact solutions we may check the physical content of the potential, compare with local potentials, fit experimental data and possibly check conclusions of general properties of such potentials. For example the spectrum gives an analytic expression of the Regge trajectory, which determines the analytic structure of the S matrix in the complex angular momentum plane.

The most interesting case for which the Schrödinger equation has dynamical symmetry is the Coulomb potential (Pauli 1926, Fock 1935, Bargmann 1936, Bander and Itzykson 1966a, b). Generalization to systems with the same symmetry leads necessarily to non-local potentials, which have been investigated in the bound-state region (Luming and Predazzi 1966a, b). In this paper we investigate the problem of O(1, f) symmetric non-local potentials in the scattering region where $f = 2, 3, 4, \ldots$. The investigation of the scattering problem is interesting from the physical as well as from the mathematical point of view. Indeed we are led to the introduction of generalized Mehler and Fourier transforms and to an extension of the known Funk–Hecke theorem for the compact $O(1+f)$ groups, to the case of the non-compact groups $O(1, f)$.

In § 2 we review the stereographic projection of the f-dimensional Schrödinger equation in momentum space on the surface of an $(f+1)$-dimensional unit sphere for negative energies (Fock 1935, Luming and Predazzi 1966a, b, Kyriakopoulos 1968). Also in the case of positive energy we project on the surface of a unit two-sheeted hyperboloid in $f+1$ dimensions. In § 3 we review the properties of the $O(1, f)$ harmonics (Bander and Itzykson 1966a, b). In § 4 we investigate the generalized Mehler and Fourier transforms. In § 5 we establish the Funk–Hecke theorem for $O(1, f)$ groups. Finally, in § 6 we solve exactly the Schrödinger equation in the scattering region, giving the analytic expressions for the energy spectrum and the wavefunctions.

2. Fock transformation

The Schrödinger equation in the f-dimensional momentum space, for an arbitrary potential, which is in general non-local and energy dependent is

$$(p^2-2\mu E)\Phi(\boldsymbol{p})=\frac{2\mu}{\hbar^f}\int \mathrm{d}^f\boldsymbol{q}\, V_{\mathrm{E}}(\boldsymbol{p}, \boldsymbol{q})\Phi(\boldsymbol{q}), \tag{2.1}$$

where

$$V_{\mathrm{E}}(\boldsymbol{p}, \boldsymbol{q})=\frac{1}{(2\pi)^f}\int \mathrm{d}^f\boldsymbol{x}\int \mathrm{d}^f\boldsymbol{x}' \exp[\mathrm{i}(-\boldsymbol{px}+\boldsymbol{qx}')/\hbar]\, V_{\mathrm{E}}(\boldsymbol{x}, \boldsymbol{x}') \tag{2.2}$$

and $V_{\mathrm{E}}(\boldsymbol{x}, \boldsymbol{x}')$ is the potential in position space. In the bound-state region we have $E<0$. Changing variables from $\boldsymbol{p}$ to $(-2\mu E)^{-1/2}\boldsymbol{p}$ and letting

$$\Phi[(-2\mu E)^{1/2}\boldsymbol{p}]=\Psi(\boldsymbol{p}) \tag{2.3}$$

$$V_{\mathrm{E}}[(-2\mu E)^{1/2}\boldsymbol{p}, (-2\mu E)^{1/2}\boldsymbol{q}]=U_{\mathrm{E}}(\boldsymbol{p}, \boldsymbol{q}), \tag{2.4}$$

equation (2.1) becomes

$$(p^2+1)\Psi(\boldsymbol{p})=\frac{(2\mu)^{f/2}E^{(f-2)/2}}{\hbar^f}\int \mathrm{d}^f\boldsymbol{q}\, U_{\mathrm{E}}(\boldsymbol{p}, \boldsymbol{q})\Psi(\boldsymbol{q}). \tag{2.5}$$

To project the f-dimensional space on the surface of an $(f+1)$-dimensional unit sphere we introduce the Fock variables

$$u_0=\frac{p^2-1}{p^2+1}, \qquad \boldsymbol{u}=\frac{2\boldsymbol{p}}{p^2+1}, \qquad u_0^2+\boldsymbol{u}^2=1. \tag{2.6}$$

In these variables equation (2.5) becomes

$$\hat{\Psi}(u)=\frac{1}{2E}\left(\frac{2\mu E}{\hbar^2}\right)^{f/2}\int \mathrm{d}^f\Omega(v) F_{\mathrm{E}}(u, v)\hat{\Psi}(v), \tag{2.7}$$

where

$$\hat{\Psi}(u)=(1-u_0)^{-(f+1)/2}\Psi(\boldsymbol{p}), \tag{2.8}$$

$$F_{\mathrm{E}}(u, v)=[(1-u_0)(1-v_0)]^{-(f-1)/2}U_{\mathrm{E}}(\boldsymbol{p}, \boldsymbol{q}) \tag{2.9}$$

and the integration in equation (2.7) takes place on the whole surface of the unit sphere.

$O(1+f)$ symmetry of expression (2.7) requires $F_E(u, v)$ to be a function of the Euclidean scalar product $uv = \cos\omega$, in which case the Hermiticity of the Hamiltonian expressed by $U_E^*(\boldsymbol{p}, \boldsymbol{q}) = U_E(-\boldsymbol{q}, -\boldsymbol{p})$, demands $F_E(u, v)$ to be real.

In this case the solution of equation (2.7) is known (Luming and Predazzi 1966a, b). If $|F_E(\cos\omega)|$ and $|F_E(\cos\omega)|^2$ are Lebesque-integrable for $-1 \leqslant \cos\omega \leqslant 1$ the eigenfunctions are the $(f+1)$-dimensional spherical harmonics $Y_{n,\alpha}|\Omega(u)|$ of degree n, and the spectrum is (Erdélyi *et al* 1953, vol 2, pp 247–8)

$$2E\left(\frac{\hbar^2}{2\mu E}\right)^{f/2} = \frac{(4\pi)^{(f-1)/2}(n-1)!\Gamma((f-1)/2)}{(n+f-3)!}\int_{-1}^{1} F_E(\cos\omega)C_{n-1}^{(f-1)/2}(\cos\omega)$$
$$\times(\sin\omega)^{f-2}\,\mathrm{d}(\cos\omega). \tag{2.10}$$

In the scattering region we have $E > 0$. We change variables from $\boldsymbol{p}$ to $(2\mu E)^{-1/2}\boldsymbol{p}$ and define

$$\Phi[(2\mu E)^{1/2}\boldsymbol{p}] = \Psi(\boldsymbol{p}), \tag{2.11}$$

$$V_E[(2\mu E)^{1/2}\boldsymbol{p}, (2\mu E)^{1/2}\boldsymbol{q}] = U_E(\boldsymbol{p}, \boldsymbol{q}). \tag{2.12}$$

In this case equation (2.1) takes the form

$$(p^2-1)\Psi(\boldsymbol{p}) = \frac{(2\mu)^{1/2}E^{(f-2)/2}}{\hbar^f}\int \mathrm{d}^f\boldsymbol{q}\, U_E(\boldsymbol{p}, \boldsymbol{q})\Psi(\boldsymbol{q}). \tag{2.13}$$

The above equation in the Fock variables

$$u_0 = \frac{1+p^2}{1-p^2}, \qquad \boldsymbol{u} = \frac{2\boldsymbol{p}}{1-p^2}, \qquad u_0^2 - \boldsymbol{u}^2 = 1 \tag{2.14}$$

takes the form

$$\hat{\Psi}(u) = \frac{\epsilon(u_0)}{2E}\left(\frac{2\mu E}{\hbar^2}\right)^{f/2}\int_{T_{f+1}} \mathrm{d}^f\mu(v)F_E(u, v)\hat{\Psi}(v), \tag{2.15}$$

where

$$\hat{\Psi}(u) = |1+u_0|^{-\frac{1}{2}(f+1)}\,\Psi(\boldsymbol{p}), \tag{2.16}$$

$$\epsilon(u_0) = \begin{cases} 1 & \text{if } u_0 > 0 \\ -1 & \text{if } u_0 < 0 \end{cases} \tag{2.17}$$

$$\mathrm{d}^f\boldsymbol{q} = \mathrm{d}^f\mu(v)/|1+v_0|^f, \tag{2.18}$$

$$F_E(u, v) = (|1+u_0|\,|1+v_0|)^{-(f-1)/2}\,U_E(\boldsymbol{p}, \boldsymbol{q}), \tag{2.19}$$

and the integration takes place on both sheets of the unit hyperboloid in the $(f+1)$-dimensional space. Equation (2.15) is $O(1, f)$ symmetric if and only if the 'potential' $F_E(u, v)$ is a function of the Minkowski scalar product $uv = u_0v_0 - \boldsymbol{u}\cdot\boldsymbol{v} = \cosh\theta$, $0 \leqslant \theta < +\infty$. Hermiticity of the Hamiltonian requires $F_E(\cosh\theta)$ to be real.

3. 'Spherical harmonics' on the hyperboloid

Let $T_{f+1}^+(T_{f+1}^-)$ be the upper (lower) sheet of the hyperboloid T_{f+1}.

$$u_0^2 - \boldsymbol{u}_i\boldsymbol{u}_i = 1, \qquad i = 1, 2 \ldots f. \tag{3.1}$$

It is known (Bander and Itzykson 1966b) that the Hilbert space of square integrable functions defined on T^+_{f+1} is the carrier space of a reducible unitary representation

$$g \to [U(\Lambda)g](u) = g(\Lambda^{-1}u),$$

of the component of the identity of the group $O(1,f)$. The spherical functions

$$\mathrm{H}^{(f+1)}_{N,\alpha,\beta}(u) = \mathrm{Z}^{(f)}_{N,\alpha}(\theta)\mathrm{Y}^{(f)}_{\alpha,\beta}(\boldsymbol{n}) \tag{3.2}$$

form a complete set of functions for the carrier space of the representation, where the real and positive parameter N classifies the irreducible representations of the group. The functions $\mathrm{Y}^{(f)}_{\alpha,\beta}(\boldsymbol{n})$ are the f-dimensional spherical harmonics and the functions $\mathrm{Z}^{(f)}_{N,\alpha}(\theta)$ are defined by

$$\mathrm{Z}^{(f)}_{N,\alpha}(\theta) = \{\tfrac{1}{2}\pi N^2(N^2+1^2)\dots[N^2+(f-1+a)^2]\}^{-1/2}(\sinh\theta)^\alpha\left(\frac{\mathrm{d}}{\mathrm{d}\cosh\theta}\right)^{\frac{1}{2}(f-1)+\alpha}\cos N\theta \tag{3.3}$$

for f odd ($f \geqslant 3$),

$$\mathrm{Z}^{(f)}_{N,\alpha}(\theta) = \left(\frac{[N^2+(\frac{1}{2})^2][N^2+(\frac{3}{2})^2]\dots\{N^2+\frac{1}{2}[(f-2)+a-\frac{1}{2}]^2\}}{N\tanh\pi N}\right)^{-1/2}(\sinh\theta)^\alpha \times\left(\frac{\mathrm{d}}{\mathrm{d}\cosh\theta}\right)^{\frac{1}{2}(f-2)+\alpha}\mathrm{P}_{\mathrm{i}N-\frac{1}{2}}(\cosh\theta) \tag{3.4}$$

for f even ($f \geqslant 2$), and $\mathrm{P}_{\mathrm{i}N-\frac{1}{2}}(\cosh\theta)$ are the conical functions (Erdélyi *et al* 1953, vol 1).

The spherical functions satisfy the orthogonality relations (Bander and Itzykson 1966b)

$$\int_{T^+_{f+1}} \mathrm{d}^f\mu(u)\mathrm{H}^{(f+1)}_{N_1,\alpha_1,\beta_1}(u)[\mathrm{H}^{(f+1)}_{N_2,\alpha_2,\beta_2}(u)]^* = \delta_{\alpha_1,\alpha_2,}\delta_{\beta_1,\beta_2}\delta(N_1-N_2) \tag{3.5}$$

and the completeness relation on the carrier space

$$\int_0^\infty \mathrm{d}N\{\Sigma_{\alpha,\beta}\mathrm{H}^{(f+1)}_{N,\alpha,\beta}(u_1)[\mathrm{H}^{(f+1)}_{N,\alpha,\beta}(u_2)]^*\} = \delta_{\mathrm{hyp}}(u_1, u_2). \tag{3.6}$$

Also these functions satisfy the addition theorems (Bander and Itzykson 1966b)

$$\Sigma_{\alpha,\beta}\mathrm{H}^{(f+1)}_{N,\alpha,\beta}(u_1)[\mathrm{H}^{(f+1)}_{N,\alpha,\beta}(u_2)]^* = \frac{(-1)^{\frac{1}{2}(f-2)}}{(2\pi)^{f/2}}N\tanh\pi N\left(\frac{\mathrm{d}}{\mathrm{d}\cosh\theta}\right)^{\frac{1}{2}(f-2)}\mathrm{P}_{\mathrm{i}N-\frac{1}{2}}(\cosh\theta) \tag{3.7}$$

for f even, and

$$\Sigma_{\alpha,\beta}\mathrm{H}^{(f+1)}_{N,\alpha,\beta}(u_1)[\mathrm{H}^{(f+1)}_{N,\alpha,\beta}(u_2)]^* = \frac{(-1)^{\frac{1}{2}(f-1)}}{\pi(2\pi)^{\frac{1}{2}(f-1)}}\left(\frac{\mathrm{d}}{\mathrm{d}\cosh\theta}\right)^{\frac{1}{2}(f-1)}\cos N\theta \tag{3.8}$$

for f odd. The functions $\mathrm{P}_{\mathrm{i}N-\frac{1}{2}}(\cosh\theta)$ are the regular, at $\cosh\theta = 1$, conical functions, with $\mathrm{P}_{\mathrm{i}N-\frac{1}{2}}(1) = 1$.

4. Generalized Mehler and Fourier transforms

To generalize the Funk–Hecke theorem (Erdélyi *et al* 1953) for $O(1,f)$ groups we introduce generalized Mehler and Fourier transforms for the cases f even and f odd respectively.

4.1. f even

Consider the class of functions $F(\cosh\theta)$ for which there exist an $\tilde F(N)$ such that

$$F(\cosh\theta)=A_f\int_0^\infty \mathrm{d}N\,\tilde F(N)\left(\frac{\mathrm{d}}{\mathrm{d}\cosh\theta}\right)^m \mathrm{P}_{\mathrm{i}N-\frac12}(\cosh\theta)N\tanh\pi N,$$
$$A_f=\frac{(-1)^{\frac12(f-2)}}{(2\pi)^{f/2}},\qquad m=\tfrac12(f-2),\tag{4.1}$$

which we call generalized Mehler transform. The conical functions $\mathrm{P}_{\mathrm{i}N-\frac12}(\cosh\theta)$ satisfy the relations

$$\int_0^\infty \mathrm{d}N\mathrm{P}_{\mathrm{i}N-\frac12}(x_1)\mathrm{P}_{\mathrm{i}N-\frac12}(x_2)N\tanh\pi N=\delta(x_1-x_2),\tag{4.2}$$

$$\int_1^\infty \mathrm{d}x\mathrm{P}_{\mathrm{i}N_1-\frac12}(x)\mathrm{P}_{\mathrm{i}N_2-\frac12}(x)=\frac{\delta(N_1-N_2)}{N_1\tanh\pi N_1}.\tag{4.3}$$

To invert equation (4.1) and calculate $\tilde F(N)$ we consider the functions

$$\tau_p(N,\theta)=(\sinh\theta)^p\left(\frac{\mathrm{d}}{\mathrm{d}\cosh\theta}\right)^p\mathrm{P}_{\mathrm{i}N-\frac12}(\cosh\theta),\qquad p=0,1,2,\ldots,\tag{4.4}$$

which satisfy the orthogonality relations (Bander and Itzykson 1966b)

$$\int_0^\infty \mathrm{d}\theta\sinh\theta\tau_p(N_1,\theta)\tau_p(N_2,\theta)=\Lambda(N_1,p)\frac{\delta(N_1-N_2)}{N_1\tanh\pi N_1},$$
$$\Lambda(N,p)=[N^2+(\tfrac12)^2][N^2+(\tfrac32)^2]\ldots[N^2+(p-\tfrac12)^2].\tag{4.5}$$

From equations (4.1) and (4.5) we get

$$\tilde F(N)=\frac{1}{A_f\Lambda(N,m)}\int_0^\infty \mathrm{d}\theta(\sinh\theta)^{2m+1}F(\cosh\theta)\left(\frac{\mathrm{d}}{\mathrm{d}\cosh\theta}\right)^m\mathrm{P}_{\mathrm{i}N-\frac12}(\cosh\theta).\tag{4.6}$$

The correspondence between the functions $F(\cosh\theta)$ and $\tilde F(N)$ is one to one if by replacing $\tilde F(N)$ in the right-hand side of equation (4.1) by its expression (4.6) we get back the function $F(\cosh\theta)$. This replacement gives

$$\int_0^\infty \mathrm{d}\theta'(\sinh\theta')^{m+1}F(\cosh\theta')\left(\frac{\mathrm{d}}{\mathrm{d}\cosh\theta}\right)^m\int_0^\infty \mathrm{d}N\tau_m(N,\theta')\tau_0(N,\theta)\frac{N\tanh\pi N}{\Lambda(N,m)}.\tag{4.7}$$

It is easy to prove that the functions $\tau_p(N,\theta)$ satisfy the relations

$$\frac{\mathrm{d}}{\mathrm{d}\cosh\theta}\frac{(\sinh\theta)^p\tau_p(N,\theta)}{N^2+(p-\frac12)^2}=-(\sinh\theta)^{p-1}\tau_{p-1}(N,\theta)\qquad p=1,2,\ldots,\tag{4.8}$$

from which we get

$$\frac{1}{\Lambda(N,p)}\left(\frac{\mathrm{d}}{\mathrm{d}\cosh\theta}\right)^p(\sinh\theta)^p\tau_p(N,\theta)=(-1)^p\tau_0(N,\theta).\tag{4.9}$$

The solution of the differential equation

$$\psi^{(k)}(x)=f(x),\qquad \psi(0)=\psi'(0)=\ldots=\psi^{(k-1)}(0)=0\tag{4.10}$$

is

$$\psi(x)=\frac{1}{(k-1)!}\int_0^x (x-t)^{k-1}f(t)\,\mathrm{d}t. \tag{4.11}$$

Therefore we get from equation (4.9)

$$\frac{1}{\Lambda(N,p)}(\sinh\theta)^p\tau_p(N,\theta)=\frac{(-1)^p}{(p-1)!}\int_0^\theta \mathrm{d}\xi\,\sinh\xi(\cosh\theta-\cosh\xi)^{p-1}\tau_0(N,\xi)$$
$$p=1,2,3,\ldots. \tag{4.12}$$

From equations (4.2), (4.4) and (4.12) we get

$$\int_0^\infty \mathrm{d}N\tau_m(N,\theta')\tau_0(N,\theta)\frac{N\tanh\pi N}{\Lambda(N,m)}$$
$$=\frac{(-1)^m}{(m-1)!}(\sinh\theta')^{-m}(\cosh\theta'-\cosh\theta)^{m-1}\mathrm{H}(\cosh\theta'-\cosh\theta),$$
$$m=1,2,3,\ldots, \tag{4.13}$$

where $\mathrm{H}(x-\psi)=1$ if $x>\psi$ and $\mathrm{H}(x-\psi)=0$ if $x<\psi$. Introducing (4.13) in expression (4.7) and using the relation

$$\left(\frac{\mathrm{d}}{\mathrm{d}\cosh\theta}\right)^p(\cosh\theta'-\cosh\theta)^{p-1}\mathrm{H}(\cosh\theta'-\cosh\theta)=(-1)^p(p-1)!\delta(\cosh\theta'-\cosh\theta) \tag{4.14}$$

we easily find that expression (4.7) becomes $F(\cosh\theta)$. For $f=2$, corresponding to $m=0$, the one-to-one correspondence between the functions $F(\cosh\theta)$ and $\tilde{F}(N)$ is easily proven with the help of equations (4.2) and (4.3). Thus our proof is completed.

4.2. *f odd*

In this case we introduce the transform

$$F(\cosh\theta)=B_f\int_0^\infty \mathrm{d}N\,\tilde{F}(N)\left(\frac{\mathrm{d}}{\mathrm{d}\cosh\theta}\right)^l\cos N\theta,$$
$$B_f=\frac{(-1)^{\frac{1}{2}(f-1)}}{\pi(2\pi)^{\frac{1}{2}(f-1)}},\qquad l=\tfrac{1}{2}(f-1). \tag{4.15}$$

To invert the above equation we define the functions

$$q_p(N,\theta)=(\sinh\theta)^p\left(\frac{\mathrm{d}}{\mathrm{d}\cosh\theta}\right)^p\cos N\theta,\qquad p=0,1,2,\ldots \tag{4.16}$$

for which we have (Bander and Itzykson 1966b)

$$\int_0^\infty \mathrm{d}\theta\,q_p(N_1,\theta)q_p(N_2,\theta)=\tfrac{1}{2}\pi M(N_1,p)\delta(N_1-N_2), \tag{4.17}$$

$$M(N,p)=N^2(N^2+1^2)(N^2+2^2)\ldots[N^2+(p-1)^2],$$

$$\frac{\mathrm{d}}{\mathrm{d}\cosh\theta}\left(\frac{(\sinh\theta)^{p-1}q_p(N,\theta)}{N^2+(p-1)^2}\right)=-(\sinh\theta)^{p-2}q_{p-1}(N,\theta), \tag{4.18}$$

and also $q_p(N, 0) = 0$ if $p \geq 1$. From equation (4.18) we get

$$\frac{1}{M(N,p)}\left(\frac{\mathrm{d}}{\mathrm{d}\cosh\theta}\right)^p[(\sinh\theta)^{p-1}q_p(N,\theta)] = (-1)^p\frac{q_0(N,\theta)}{\sinh\theta}. \tag{4.19}$$

In analogy to the case f even we obtain from equation (4.19)

$$\frac{1}{M(N,p)}(\sinh\theta)^{p-1}q_p(N,\theta) = \frac{(-1)^p}{(p-1)!}\int_0^\theta \mathrm{d}\xi(\cosh\theta - \cosh\xi)^{p-1}q_0(N,\xi)$$
$$p = 1, 2, 3, \ldots. \tag{4.20}$$

From equations (4.15) and (4.17) we get

$$\tilde{F}(N) = \frac{2}{\pi B_f M(N,1)}\int_0^\infty \mathrm{d}\theta(\sinh\theta)^{2l}F(\cosh\theta)\left(\frac{\mathrm{d}}{\mathrm{d}\cosh\theta}\right)^l \cos N\theta. \tag{4.21}$$

Introducing the expression (4.21) in the right-hand side of equation (4.15) and proceeding as in the case of f even we establish the one-to-one correspondence between the functions $F(\cosh\theta)$ and $\tilde{F}(N)$.

We want to find the conditions under which the transform $\tilde{F}(N)$ exists. Consider first the case f even. Suppose that the function

$$f(\cosh\theta) = \left(\frac{\mathrm{d}}{\mathrm{d}\cosh\theta}\right)^m[(\sinh\theta)^{2m}F(\cosh\theta)]$$

satisfies the conditions:

(i) $$\int_0^\infty \frac{|f(\cosh\theta)|}{(\cosh\theta)^{1/2}}\sinh\theta\,\mathrm{d}\theta < \infty;$$

(ii) it is a function of bounded variation for θ in the interval $0 \leq \theta \leq \infty$, under which it has a Mehler transform (Robin 1959)

$$\left(\frac{\mathrm{d}}{\mathrm{d}\cosh\theta}\right)^m[(\sinh\theta)^{2m}F(\cosh\theta)] = \int_0^\infty \mathrm{d}N\tilde{f}(N)\tau_0(N,\theta)N\tanh\pi N, \tag{4.22}$$

where $\tau_0(N,\theta) = \mathrm{P}_{\mathrm{i}N-\frac{1}{2}}(\cosh\theta)$. Also we assume that

(iii) $$\left(\frac{\mathrm{d}}{\mathrm{d}\cosh\theta}\right)^k[(\sinh\theta)^{2m}F(\cosh\theta)]|_{\theta=0} = 0, \qquad k = 0, 1, 2, \ldots m-1. \tag{4.23}$$

Using equations (4.22) and (4.23) we shall prove the existence of the transform (4.1). Indeed from equations (4.10), (4.11), (4.22) and (4.23) we get

$$(\sinh\theta)^{2m}F(\cosh\theta) = \int_0^\infty \mathrm{d}N\tilde{f}(N)N\tanh\pi N\frac{1}{(m-1)!}\int_0^\theta \mathrm{d}\xi\sinh\xi$$
$$\times(\cosh\theta - \cosh\xi)^{m-1}\mathrm{P}_{\mathrm{i}N-\frac{1}{2}}(\cosh\xi). \tag{4.24}$$

Defining the function $\tilde{F}(N) = (-1)^m\tilde{f}(N)/A_f\Lambda(N,m)$ and using equation (4.12) the above equation becomes the transform (4.1). Thus we see that conditions (i), (ii) and (iii) are sufficient for the existence of the transform (4.1) and its inverse, establishing in this way a one-to-one correspondence between the set of functions which have the generalized Mehler transform of equation (4.1), and the set of functions which have the usual Mehler transform.

Consider now the case f odd. The functions $\sinh\theta(\mathrm{d}/\mathrm{d}\cosh\theta)^l \times[(\sinh\theta)^{2l-1} F(\cosh\theta)]$ have cosine Fourier transform if they are absolutely integrable in the interval $0\leq\theta\leq\infty$. Then we write

$$\sinh\theta\left(\frac{\mathrm{d}}{\mathrm{d}\cosh\theta}\right)^l[(\sinh\theta)^{2l-1}F(\cosh\theta)]=\int_0^\infty \mathrm{d}N\tilde{f}(N)q_0(N,\theta), \tag{4.25}$$

where $q_0(N,\theta)=\cos N\theta$. Also we assume that

$$\sinh\theta\left(\frac{\mathrm{d}}{\mathrm{d}\cosh\theta}\right)^k[(\sinh\theta)^{2l-1}F(\cosh\theta)]|_{\theta=0}=0,$$
$$k=0,1,2,,,,l-1. \tag{4.26}$$

We shall prove that these are sufficient conditions for the existence of the transform (4.15) and its uniform convergence. From equations (4.10), (4.11), (4.25) and (4.26) we get

$$(\sinh\theta)^{2l-1}F(\cosh\theta)=\int_0^\infty \mathrm{d}N\tilde{f}(N)\frac{1}{(l-1)!}\int_0^\theta \mathrm{d}\xi(\cosh\theta-\cosh\xi)^{l-1}q_0(N,\xi). \tag{4.27}$$

Using equation (4.20) we get the transform (4.15) where $\tilde{F}(N)=(-1)^l\tilde{f}(N)/B_fM(N,l)$. Therefore the transform (4.15) exists if the left-hand side of equation (4.25) is absolutely integrable and conditions (4.26) are satisfied.

5. Generalized Funk–Hecke theorem

We shall establish the Funk–Hecke theorem for $O(1,f)$ groups first on the upper sheet of the unit hyperboloid T^+_{f+1} and then on both sheets.

5.1. Upper sheet

From equations (3.7) and (4.1) for f even or equations (3.8) and (4.15) for f odd we get

$$F(\cosh\theta)=\int_0^\infty \mathrm{d}N\tilde{F}(N)\{\Sigma_{\alpha,\beta}\mathrm{H}^{(f+1)}_{N,\alpha,\beta}(u_1)[\mathrm{H}^{(f+1)}_{N,\alpha,\beta}(u_2)]^*\}, \tag{5.1}$$

where $\tilde{F}(N)$ is given by equations (4.6) for f even and (4.21) for f odd. From equation (5.1) and the orthogonality relations (3.5) we obtain the generalized Funk–Hecke theorem on the upper sheet of the unit hyperboloid

$$\int_{T^+_{f+1}} \mathrm{d}^f\mu(u_2)F(u_1u_2)\mathrm{H}_{N,\alpha,\beta}(u_2)=\tilde{F}(N)\mathrm{H}_{N,\alpha,\beta}(u_1), \tag{5.2}$$

where $u_1u_2=\cosh\theta$, and the measure on the upper sheet of the hyperboloid is $\mathrm{d}^f\mu(u)=\mathrm{H}(u_0)2\delta(u^2-1)\mathrm{d}^{f+1}u$.

5.2. Both sheets

Let $\mathrm{h}_{N,\alpha,\beta}(u)$ be a function defined on the whole hyperboloid $T_{f+1}=T_{f+1}^{+}\cup T_{f+1}^{-}$ by

$$\mathrm{h}_{N,\alpha,\beta}(u)=\begin{cases}\mathrm{H}_{N,\alpha,\beta}(u) & \text{if } u\in T_{f+1}^{+}\\ C\mathrm{H}_{N,\alpha,\beta}(-u) & \text{if } u\in T_{f+1}^{-}\end{cases}. \tag{5.3}$$

We can extend the Funk–Hecke theorem on both sheets if both transforms, $\tilde{F}(N)$ of $F(\cosh\theta)$ and $\tilde{G}(N)$ of $F(-\cosh\theta)$ exist. We shall compute the integral

$$\int_{T_{f+1}}\mathrm{d}^{f}\mu(u_2)F(u_1u_2)\mathrm{H}_{N,\alpha,\beta}(u_2).$$

Using (5.3) we get

$$\int_{T_{f+1}^{+}}\mathrm{d}^{f}\mu(u_2)F(u_1u_2)\mathrm{h}_{N,\alpha,\beta}(u_2)=\begin{cases}\tilde{F}(N)\mathrm{h}_{N,\alpha,\beta}(u_1) & \text{if } u_1\in T_{f+1}^{+}\\ C^{-1}\tilde{G}(N)\mathrm{h}_{N,\alpha,\beta}(u_1) & \text{if } u_1\in T_{f+1}^{-}\end{cases} \tag{5.4}$$

and

$$\int_{T_{f+1}^{-}}\mathrm{d}^{f}\mu(u_2)F(u_1u_2)\mathrm{h}_{N,\alpha,\beta}(u_2)=\begin{cases}C\tilde{G}(N)\mathrm{h}_{N,\alpha,\beta}(u_1) & \text{if } u_1\in T_{f+1}^{+}\\ \tilde{F}(N)\mathrm{h}_{N,\alpha,\beta}(u_1) & \text{if } u_1\in T_{f+1}^{-},\end{cases} \tag{5.5}$$

where the measure on the lower sheet of the hyperboloid is $\mathrm{d}^{f}\mu(u)=\mathrm{H}(-u_0)2\delta(u^2-1)\mathrm{d}^{f+1}u$. From equations (5.4) and (5.5) we obtain

$$\int_{T_{f+1}}\mathrm{d}^{f}\mu(u_2)F(u_1u_2)\mathrm{h}_{N,\alpha,\beta}(u_2)=\begin{cases}(\tilde{F}(N)+C\tilde{G}(N))\mathrm{h}_{N,\alpha,\beta}(u_1) & \text{if } u_1\in T_{f+1}^{+}\\ (\tilde{F}(N)+C^{-1}\tilde{G}(N))\mathrm{h}_{N,\alpha,\beta}(u_1) & \text{if } u_1\in T_{f+1}^{-}\end{cases}. \tag{5.6}$$

To get a Funk–Hecke theorem we must fix the constant factor C by the relation $\tilde{F}(N)+C\tilde{G}(N)=\tilde{F}(N)+C^{-1}\tilde{G}(N)$ which gives $C=\pm1$. Therefore we get

$$\int_{T_{f+1}}\mathrm{d}^{f}\mu(u_2)F(u_1u_2)\mathrm{h}_{N,\alpha,\beta}^{\pm}(u_2)=\tilde{F}^{\pm}(N)\mathrm{h}_{N,\alpha,\beta}^{\pm}(u_1), \tag{5.7}$$

where

$$\tilde{F}^{\pm}(N)=\tilde{F}(N)\pm\tilde{G}(N) \tag{5.8}$$

and

$$\mathrm{h}_{N,\alpha,\beta}^{\pm}(u)=\begin{cases}\mathrm{H}_{N,\alpha,\beta}(u) & \text{if } u\in T_{f+1}^{+}\\ \pm\mathrm{H}_{N,\alpha,\beta}(-u) & \text{if } u\in T_{f+1}^{-}\end{cases}. \tag{5.9}$$

6. Schrödinger equation for scattering

We want to solve the Schrödinger equation (2.15) if the potential $F_{\mathrm{E}}(u_1, u_2)$ has O(1, f) symmetry. To obtain an equation of this form we must impose on equation (5.6) the condition

$$\tilde{F}_{\mathrm{E}}(N)+C_{\mathrm{E}}\tilde{G}_{\mathrm{E}}(N)=-(\tilde{F}_{\mathrm{E}}(N)+C_{\mathrm{E}}^{-1}\tilde{G}_{\mathrm{E}}(N)), \tag{6.1}$$

because of the presence of the factor $\epsilon(u_0)$ in equation (2.15). Therefore we find

$$C_{\mathrm{E}}^{\pm}=\frac{-\tilde{F}_{\mathrm{E}}(N)\pm(\tilde{F}_{\mathrm{E}}^{2}(N)-\tilde{G}_{\mathrm{E}}^{2}(N))^{1/2}}{\tilde{G}_{\mathrm{E}}(N)}, \tag{6.2}$$

and we get

$$\int_{T_{f+1}} \mathrm{d}^f\mu(v) F_{\mathrm{E}}^{\pm}(uv) \mathrm{h}_{N,\alpha,\beta}^{\pm}(v) = (\tilde{F}_{\mathrm{E}}^2(N) - \tilde{G}_{\mathrm{E}}^2(N))^{1/2} \epsilon(u_0) \mathrm{h}_{N,\alpha,\beta}^{\pm}(u), \tag{6.3}$$

where

$$\mathrm{h}_{N,\alpha,\beta}^{\pm}(u) = \begin{cases} \mathrm{H}_{N,\alpha,\beta}(u) & \text{if } u \in T_{f+1}^{+} \\ C_{\mathrm{E}}^{\pm}\mathrm{H}_{N,\alpha,\beta}(-u) & \text{if } u \in T_{f+1}^{-}, \end{cases} \tag{6.4}$$

and $F_{\mathrm{E}}^{\pm}(uv) = \pm F_{\mathrm{E}}(uv)$. Comparing equations (2.15) and (6.3) we get the equation

$$\frac{1}{2E}\left(\frac{2\mu E}{\hbar^2}\right)^{f/2} = (\tilde{F}_{\mathrm{E}}^2(N) - \tilde{G}_{\mathrm{E}}^2(N))^{-1/2}, \tag{6.5}$$

from which the energy spectrum $E = E(N)$ will be determined.

As an example we consider the Schrödinger equation with Coulomb potential in f dimensions. In this case we have equation (2.1) with

$$V_{\mathrm{E}}(\boldsymbol{p}, \boldsymbol{q}) = \frac{e^2\hbar^{f-1}}{\pi\omega_{f-1}} \frac{1}{|\boldsymbol{p}-\boldsymbol{q}|^{f-1}}, \qquad \omega_{f-1} = \frac{2\pi^{(f-1)/2}}{\Gamma[(f-1)/2]}, \tag{6.6}$$

and we get

$$F_{\mathrm{E}}(uv) = \frac{e^2}{\pi\omega_{f-1}}\left(\frac{\hbar^2}{2\mu E}\right)^{(f-1)/2} \frac{1}{|(u-v)^2|^{(f-1)/2}}. \tag{6.7}$$

From equation (5.2) and the integral equation (Bander and Itzykson 1966b)

$$\int_{T_{f+1}^{+}} \mathrm{d}^f\mu(v) \frac{\mathrm{H}_{N,\alpha,\beta}^{f+1}(v)}{(1+t^2+2tuv)^{(f-1)/2}} = \frac{2\pi^{\frac{1}{2}(f+1)}}{\Gamma[(f-1)/2]} \frac{t^{-(f-1)/2}\cos(N\ln t)}{N\sinh \pi N} \mathrm{H}_{N,\alpha,\beta}^{f+1}(u), \tag{6.8}$$

the expressions $\tilde{F}_{\mathrm{E}}(N)$ and $\tilde{G}_{\mathrm{E}}(N)$ can be obtained. Taking $t = -1$ and $t = 1$ in equation (6.8) we get respectively

$$\tilde{F}_{\mathrm{E}}(N) = e^2\left(\frac{\hbar^2}{2\mu E}\right)^{(f-1)/2} \frac{\cosh \pi N}{N\sinh \pi N}, \tag{6.9}$$

$$\tilde{G}_{\mathrm{E}}(N) = e^2\left(\frac{\hbar^2}{2\mu E}\right)^{(f-1)/2} \frac{1}{N\sinh \pi N}. \tag{6.10}$$

From equations (6.5), (6.9) and (6.10) we find the well known Coulomb energy spectrum

$$E = \frac{\mu e^2}{2\hbar^2}\frac{1}{N^2}. \tag{6.11}$$

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